(1) Consider the following formal language:

- The alphabet consists of two symbols: 0 and 1 and +
- The following rules define admissible words:
  i) Empty word is admissible.
  ii) If \( \omega \) is an admissible word then so are \( \omega 1 \) and \( 0\omega \). For example, since the empty word is admissible, if we attach '0' to the left of the empty word we get '0' as an admissible word.

- The grammatically correct sentences are either admissible words, or sequences 2 or more of admissible words put together using the '+' symbol provided that the sentence contains an equal number of 0’s and 1’s. For example, both '01' and '001' are admissible words, but the sentence “01+001” is not grammatically correct because it contains three 0’s and two 1’s.

(a) Decide if the following words are admissible:

- “0101”, “0000”, “1110”, “00011111” and “110011”

   **Answer:** Admissible words are the words that start with a contiguous sequence 0’s followed by a contiguous sequence of 1’s. Therefore the only admissible words we have here are “0000” and “00011111” and “110011”. The others are inadmissible.

(b) Decide which of the following sentences are grammatically correct.

- “00+11+0011” **(Answer: Grammatically correct sentence.)**
- “1010+1001+110011” **(Answer: It is not a grammatically correct sentence because it consists of inadmissible words.)**
- “1+0+11+000+00011” **(Answer: Even though it contains admissible words, the sentence is not grammatically correct since the number of 0’s and 1’s are not equal.)**

(2) For the following question, we use the English language. Decide if the followings are sentences, propositions or neither.

(a) “Yesterday, I went to the market and bought 4 apples.”

   **Answer:** Grammatically correct sentence. A proposition.

(b) “The weather is too hot.”

   **Answer:** Grammatically correct sentence. Not a proposition.
(c) “Watch out!”
   **Answer:** Grammatically correct sentence. Not a proposition.

(d) “Chargoggagoggmanchaugagoggchaubunagungamaug.”
   **Answer:** Grammatically correct sentence, but not in English.

(e) “The temperature is 30°C.”
   **Answer:** Grammatically correct sentence. A proposition.

(f) “X is a 40 year old man.”
   **Answer:** Grammatically correct sentence. Not a proposition.

(g) “I have never seen him in my life.”
   **Answer:** Grammatically correct sentence. A proposition.

(h) “This sentence is a false proposition.”
   **Answer:** Grammatically correct sentence. Not a proposition.

(i) “Anyone older than 30 years is a liar.”
   **Answer:** Grammatically correct sentence. A proposition.

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(3) Write all instances of the following propositional functions and determine their truth values

(a) P(x): “x is a day of the week.” where x is an element of the set {monday, tuesday, april, september, monkey}.
   **Answer:**
   (i) P(Monday): “Monday is a day of the week.” TRUE
   (ii) P(Tuesday): “Tuesday is a day of the week.” TRUE
   (iii) P(April): “April is a day of the week.” FALSE
   (iv) P(September): “September is a day of the week.” FALSE
   (v) P(Monkey): “Monkey is a day of the week.” FALSE

(b) Q(x): “x is a letter in the English alphabet and x comes before the letter ‘f’ in the alphabetical order.” where x is an element of the set {a,b,c,g,x,y,α,ℵ}.
   **Answer:**
   (i) Q(a): “‘a’ is a letter in the English alphabet and ‘a’ comes before the letter ‘f’ in the alphabetical order.” (TRUE)
   (ii) Q(b): “‘b’ is a letter in the English alphabet and ‘b’ comes before the letter ‘f’ in the alphabetical order.” (TRUE)
   (iii) Q(c): “‘c’ is a letter in the English alphabet and ‘c’ comes before the letter ‘f’ in the alphabetical order.” (TRUE)
   (iv) Q(g): “‘g’ is a letter in the English alphabet and ‘g’ comes before the letter ‘f’ in the alphabetical order.” (FALSE)
(v) $Q(x):$ ‘$x$’ is a letter in the English alphabet and ‘$x$’ comes before the letter ‘$f$’ in the alphabetical order.” (FALSE)

(vi) $Q(y):$ ‘$y$’ is a letter in the English alphabet and ‘$y$’ comes before the letter ‘$f$’ in the alphabetical order.” (FALSE)

(vii) $Q(\alpha):$ ‘$\alpha$’ is a letter in the English alphabet and ‘$\alpha$’ comes before the letter ‘$f$’ in the alphabetical order.” (FALSE)

(viii) $Q(\aleph):$ ‘$\aleph$’ is a letter in the English alphabet and ‘$\aleph$’ comes before the letter ‘$f$’ in the alphabetical order.” (FALSE)

(c) Let $A = \{1, 2, 3, 5, 7\}$ and consider the propositional function

$R(x,t):$ “There are exactly $t$ number of elements in the sequence 1,2,...,$x$ from the set $A.$”

where $t \in \{0, 1, 2\}$ and $x \in A.$

**Answer:** There are 15 instances of the function one for each of the element of the product set

$$\{0, 1, 2\} \times \{1, 2, 3, 5, 7\} = \{(1, 0), (2, 0), (3, 0), (5, 0), (7, 0),$$

$$(1, 1), (2, 1), (3, 1), (5, 1), (7, 1),$$

$$(1, 2), (2, 2), (3, 2), (5, 2), (7, 2)\}$$

I will write only one here. You should write the rest and test.

$R(3,2):$ “There are exactly 2 elements in the sequence 1,2,3 from the set $A.$” (FALSE)

(4) Consider the propositional function:

$P(\alpha, \beta):$ “$\alpha$ comes before $\beta$ in the alphabetical order.”

where $\alpha, \beta \in A = \{a,b,c,d\}.$ Expand the following propositions in terms of basic logical operations of conjunctions and disjunctions. After the expansion, evaluate the truth value of the resulting sentence.

(a) $\forall \alpha \in A, \forall \beta \in A, \ P(\alpha, \beta)$

**Answer:**

$$(P(a,a) \land P(a,b) \land P(a,c) \land P(a,d))$$

$\land (P(b,a) \land P(b,b) \land P(b,c) \land P(b,d))$

$\land (P(c,a) \land P(c,b) \land P(c,c) \land P(c,d))$

$\land (P(d,a) \land P(d,b) \land P(d,c) \land P(d,d)) \equiv \text{FALSE}$$
(b) $\forall \alpha \in A, \exists \beta \in A, P(\alpha, \beta)$

**Answer:**

$$(P(a, a) \lor P(a, b) \lor P(a, c) \lor P(a, d))$$
$$\land (P(b, a) \lor P(b, b) \lor P(b, c) \lor P(b, d))$$
$$\land (P(c, a) \lor P(c, b) \lor P(c, c) \lor P(c, d))$$
$$\land (P(d, a) \lor P(d, b) \lor P(d, c) \lor P(d, d)) \equiv \text{TRUE}$$

(c) $\exists \alpha \in A, \forall \beta \in A, P(\alpha, \beta)$

**Answer:**

$$(P(a, a) \land P(a, b) \land P(a, c) \land P(a, d))$$
$$\lor (P(b, a) \land P(b, b) \land P(b, c) \land P(b, d))$$
$$\lor (P(c, a) \land P(c, b) \land P(c, c) \land P(c, d))$$
$$\lor (P(d, a) \land P(d, b) \land P(d, c) \land P(d, d)) \equiv \text{TRUE}$$

(d) $\exists \alpha \in A, \exists \beta \in A, P(\alpha, \beta)$

**Answer:**

$$(P(a, a) \lor P(a, b) \lor P(a, c) \lor P(a, d))$$
$$\lor (P(b, a) \lor P(b, b) \lor P(b, c) \lor P(b, d))$$
$$\lor (P(c, a) \lor P(c, b) \lor P(c, c) \lor P(c, d))$$
$$\lor (P(d, a) \lor P(d, b) \lor P(d, c) \lor P(d, d)) \equiv \text{TRUE}$$

(5) Consider the set $X = \{-1, 0, 1\}$ and the predicates

$$P(x) : x^2 = x$$
$$Q(x) : x \geq 0$$

where we allow only $x \in X$.

(a) Write the sets $A = \{x \in X| P(x)\}$ and $B = \{x \in X| Q(x)\}$.

**Answer:** Both sets are the same $\{0, 1\}$.

(b) Verify that $P(x) \equiv Q(x)$.

**Answer:** The predicates are equivalent because the sets defined by these predicates are equal.
(6) Consider the alphabet \( A = \{0, 1, 2\} \) with the order \( 0 < 1 < 2 \). Consider the predicates

\[
Q(x) : x \text{ has length greater than 3} \\
P(x) : x \text{ comes before the word "011" in the lexicographical order}
\]

where the variable \( x \) can only take values in the set of words in \( A \). Write the elements of the set

\[
\{ x \mid \neg P(x) \land Q(x) \}
\]

**Answer:** The negation of \( P(x) \) is equivalent to the proposition

\[
S(x) : x \text{ has length 3 or less}
\]

So the set we are looking for is

\[
\{ x \mid x \text{ has length 3 or less and } x \text{ comes before the word "011" in the lexicographical order} \}
\]

That set is

\[
\{0, 00, 01, 000, 001, 002, 010, 011\}
\]

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(7) Let \( A = \{1, 2, 4, 9\} \) and let \( P(x): \exists y \in A, y^2 = x \).

(a) Evaluate the truth value of \( P(1) \).

**Answer:** \( P(1) : \exists y \in \{1, 2, 4, 9\}, y^2 = 1 \) \( \equiv \) TRUE

(b) Write the negation of \( P(2) \). Evaluate its truth value.

**Answer:** \( P(2) : \exists y \in \{1, 2, 4, 9\}, y^2 = 2 \) \( \equiv \) FALSE

(c) Write the proposition \( \forall x \in A, P(x) \) and evaluate its truth value.

**Answer:** \( \forall x \in A, P(x) \equiv P(1) \land P(2) \land P(4) \land P(9) \equiv \) FALSE

(d) Using De Morgan’s Laws, expand

\[
\neg (\exists x \in A, P(x))
\]

and evaluate its truth value.

**Answer:**

\[
\neg (\exists x \in A, P(x)) \equiv \forall x \in A, \neg P(x) \equiv \neg P(1) \land \neg P(2) \land \neg P(4) \land \neg P(9) \equiv \text{False}
\]

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(8) Let \( \text{Prop} \) be the set of all propositions. Give a convincing argument why

\[
\forall P \in \text{Prop}, P \implies (P \lor \neg P)
\]

is true. On the other hand

\[
\forall P \in \text{Prop}, (P \lor \neg P) \implies P
\]
is not true. Explain why.

Answer: The truth values of the propositional functions “$P \implies (P \lor \neg P)$” and “$(P \lor \neg P) \implies P$” depend entirely on the truth value of $P$. There are two cases:

(a) $P$ is true: then “$P \implies (P \lor \neg P)$” is also True.
(b) $P$ is false: then “$P \implies (P \lor \neg P)$” is True.

Then “$P \implies (P \lor \neg P)$” is true regardless of the truth value of $P$. Thus we conclude $\forall P \in \text{Prop}$, “$P \implies (P \lor \neg P)$” is also True. Now, repeat a similar argument for the other propositional function.

(9) Consider the finite set of sequences of 0’s and 1’s $A = \{1, 10, 11, 100, 110\}$

(a) Consider the relation $R: A \to \mathbb{N}$ given by

$$R = \{(a, n) \in A \times \mathbb{N} | \text{ the number of 1’s in the sequence } a \text{ is greater than } n\}$$

Write the graph of the relation $R$. In other words, write $R$ as a subset of $A \times \mathbb{N}$ explicitly in terms of its elements.

Answer: The number $n$ in any pair $(a, n)$ must be greater than the number of 1’s in the sequence $a \in A$. For example $(1, 100) \in R$ but $(1, 0) \notin R$. Also $(110, 3) \in R$ but the elements $(110, 0), (110, 1)$ and $(110, 2)$ are not in $R$.

$$R = \{(1, 2), (1, 3), (1, 4), \ldots \}$$

$$\cup \{(10, 2), (10, 3), (10, 4), \ldots \}$$

$$\cup \{(11, 3), (11, 4), (11, 5), \ldots \}$$

$$\cup \{(100, 2), (100, 3), (100, 4), \ldots \}$$

$$\cup \{(110, 3), (110, 4), (110, 5), \ldots \}$$

(b) Consider the relation $S: A \to A$ given by

$$S = \{(x, y) \in A \times A | \text{ the lengths of } x \text{ and } y \text{ are the same}\}$$

Write the graph of $S$. Then sketch the Venn diagram representation, and the directed graph representation of $S$.

Answer: I will write the graph only. You should do the pictures.

$$\{(1, 1), (10, 10), (10, 11), (11, 10), (11, 11), (100, 100), (100, 110), (110, 100), (110, 110)\}$$

(c) For the relation in part (b) above, write the graph of the reverse relation $S^{op}$ and the composition relation $S \circ S^{op}$.

Answer: This relation is an equivalence relation. In particular, it is symmetric. So
$S^{op} = S$. This also means $S \circ S^{op} = S \circ S$. Moreover, $S$ is also transitive. Therefore $S \circ S \subseteq S$ and since $S$ is also reflexive $S \circ S = S$.

(10) Consider the relation $f: A \rightarrow A$ where $A = \{0, 1, 2, 3, 4\}$ and

$$f(x) = \text{the first even number that comes before } x \text{ in } A$$

(a) Write the graph of $f$.

\textbf{Answer:} $\{(1, 0), (2, 0), (3, 0), (4, 0)\}$

(b) Sketch the directed graph of $f$.

(c) Verify that $f$ is a partial function.

\textbf{Answer:} We must verify that

$$\forall x, y, z \in A, (x, y) \in f \land (x, z) \in f \implies y = z$$

Since $A$ contains 5 elements, the number of checks we must perform is 125. This is too much. I will take a short-cut. Remember that an implication $P \implies Q$ is true when the proposition $P$ is false. So, if we are looking for a false instance of “$(x, y) \in f \land (x, z) \in f \implies y = z$” we must concentrate on the cases where “$(x, y) \in f \land (x, z) \in f$” is true and those are

(i) “$(1, 0) \in f \land (1, 0) \in f$”

(ii) “$(2, 0) \in f \land (2, 0) \in f$”

(iii) “$(3, 0) \in f \land (3, 0) \in f$”

(iv) “$(4, 0) \in f \land (4, 0) \in f$”

and in all of these cases “$y = z$” is true. This means the implication “$(x, y) \in f \land (x, z) \in f \implies y = z$” is never false in all of the 125 instances.

(d) Verify that $f$ is not a function.

\textbf{Answer:} I will verify that the following proposition is false

$$\forall x \in A, \exists y \in A, (x, y) \in f$$

There are 25 instances of the propositional function $(x, y) \in f$. Again, this is too many. So consider the following propositional function

$$P(x): \exists y \in A, (x, y) \in f$$

and our original proposition “$\forall x \in A, \exists y \in A, (x, y) \in f$” is the same as “$\forall x \in A, P(x)$.” Since this last proposition comes out from a universal quantifier, in
order to show that it is false, we need a false instance of \( P(x) \). This is easy since
\[
P(0) : \exists y \in \{0, 1, 2, 3, 4\} \quad (0, y) \in f
\]
\[
\equiv (0, 0) \in f \lor (0, 1) \in f \lor (0, 2) \in f \lor (0, 3) \in f \lor (0, 4) \in f
\]
\[
\equiv \text{FALSE}
\]

(11) Consider the relation \( f : \mathbb{N} \to A \) where \( A = \{0, 2, 4, 6, 8\} \) and \( f \) is defined as
\[
f = \{(n, a) \in \mathbb{N} \times A | \text{ } n \text{ is an even integer and the last digit of the number } n \text{ in its decimal expansion is } a \}
\]
For example \((10, 0)\) is in \( f \) but \((10, 2)\) is not in \( f \). Verify that \( f \) is a partial function. Verify that \( f \) is not a function.

**Answer:** In order \( f \) to be a partial function, the following proposition must be true:
\[
\forall x \in \mathbb{N}, \forall y, z \in \{0, 2, 4, 6, 8\} \quad (x, y) \in f \land (x, z) \in f \implies y = z
\]
There are infinitely many instances of the propositional function \( \forall x \in \mathbb{N}, \exists y \in \{0, 2, 4, 6, 8\} \quad (x, y) \in f \) because \( x \) comes from an infinite set. So, I will use the same short-cut I used before: Try to find a false instance of the propositional function above. Since we are looking for a false example, concentrate on the cases where \( \forall x \in \mathbb{N}, \exists y \in \{0, 2, 4, 6, 8\} \quad (x, y) \in f \) is true. So, \( x \) is an even integer and \( y \) is the last digit of the decimal expansion of \( x \), and \( z \) is also the last digit of the decimal expansion of \( x \). But \( x \) has only one decimal expansion. So, \( y \) and \( z \) must be the same number. This means \( y = z \) is true in all cases when \( \forall x \in \mathbb{N}, \exists y \in \{0, 2, 4, 6, 8\} \quad (x, y) \in f \) is true. This means
\[
(x, y) \in f \land (x, z) \in f \implies y = z
\]
can not be false. Then
\[
\forall x \in \mathbb{N}, \forall y, z \in \{0, 2, 4, 6, 8\} \quad (x, y) \in f \land (x, z) \in f \implies y = z
\]
is true. To check that \( f \) is a not a function, we must evaluate the truth value of the proposition
\[
\forall x \in \mathbb{N}, \exists y \in \{0, 2, 4, 6\} \quad (x, y) \in f
\]
Again, we can not perform an infinite number of checks. Consider the propositional function
\[
P(x) : \exists y \in \{0, 2, 4, 6\} \quad (x, y) \in f
\]
where \( x \in \mathbb{N} \). Then we also have
\[
(\forall x \in \mathbb{N}, \exists y \in \{0, 2, 4, 6\} \quad (x, y) \in f) \equiv (\forall x \in \mathbb{N}, \ P(x))
\]
And since there is a universal quantifier, it is easy to verify that it is false. We need a false example.

\[ P(1) : \exists y \in \{0, 2, 4, 6, 8\}, \ (1, y) \in f \]
\[ \equiv (1, 0) \in f \lor (1, 2) \in f \lor (1, 4) \in f \lor (1, 6) \in f \lor (1, 8) \in f \]
\[ \equiv \text{FALSE} \]

(12) Consider the set \( A = \{0, 1, 2, 3, 4, 5\} \) and the relation \( R: A \rightarrow A \) defined as

\[ R = \{(a, b) \in A \times A | a + b \text{ is even}\} \]

Sketch the directed graph of \( R \), and decide if \( R \) is reflexive, symmetric, anti-symmetric, and transitive. Is it an equivalence relation, or a partial order relation?

**Answer:** The elements of the relation can be written as

\[ R = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (0, 2), (0, 4), (2, 4),
(2, 0), (4, 0), (4, 2), (1, 3), (1, 5), (3, 5), (3, 1), (5, 1), (5, 3)\} \]

The relation is reflexive, symmetric, and transitive, but not anti-symmetric. So, \( R \) is an equivalence relation, but not an partial order relation.

(13) Consider the set \( A = \{0, 1, 2\} \) and the relation

\[ E = \{(0, 0), (0, 1), (1, 2), (2, 1)\} \]

(a) Decide if the relation is reflexive, symmetric, anti-symmetric or transitive.

**Answer:** \( E \) is not reflexive, not symmetric, not anti-symmetric, not transitive.

(b) Which edges should we add to make \( E \) into a reflexive relation?

**Answer:** We need to add the edges \((1, 1)\) and \((2, 2)\).

(c) Which edges should we add to make \( E \) into a symmetric relation?

**Answer:** We need to add \((1, 0)\).

(d) Which edge(s) should we remove to make \( E \) into a anti-symmetric relation?

**Answer:** The anti-symmetric relations we can obtain from \( E \) by removing edges are

(i) The empty relation \(\emptyset\)

(ii) Four relations each containing a single edge

\[ E_1 = \{(0, 0)\} \quad E_2 = \{(0, 1)\} \quad E_3 = \{(1, 2)\} \quad E_4 = \{(2, 1)\} \]
(iii) 5 relations each containing 2 edges

\[ E_5 = \{(0, 0), (0, 1)\} \quad E_6 = \{(0, 0), (1, 2)\} \quad E_7 = \{(0, 0), (2, 1)\} \]
\[ E_8 = \{(0, 1), (1, 2)\} \quad E_9 = \{(0, 1), (2, 1)\} \]

(iv) 2 relations each containing 3 elements

\[ E_{10} = \{(0, 0), (0, 1), (1, 2)\} \quad E_{11} = \{(0, 0), (0, 1), (2, 1)\} \]

(e) Which edge(s) should we add to make \( E \) into a transitive relation?

**Answer:** We must add the edges \((0, 2), (1, 1)\) and \((2, 2)\).